A Note on Optimal Algorithms for Fixed Points

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Abstract

We present a constructive lemma that we believe will make possible the design of nearly optimal $O(d \log \frac{1}{\epsilon})$ cost algorithms for computing ϵ-residual approximations to the fixed points of d-dimensional nonexpansive mappings with respect to the infinity norm. This lemma is a generalization of a two-dimensional result that we proved in [1].

1 Introduction

In [1, 2] we presented two-dimensional optimal complexity algorithms for computing residual ϵ-approximations to the fixed points of non-expansive mappings with respect to the infinity norm. These algorithms are based on bisection-envelope constructions and are derived from Theorem 3.1 of [1]. This theorem makes possible construction of a sequence of rectangles that contain fixed points and converge to the residual ϵ-approximation of some fixed point. At every iteration of the process the previous rectangle is cut by a factor of at least two, to obtain a new rectangle containing a fixed point.

In this paper we generalize the constructive theorem to an arbitrary number of dimensions $d \geq 3$, however, we are unable to utilize this new result in the construction of optimal algorithms.

The main obstacle in such construction is the ability to bound a new set containing fixed points by an "easy-to-construct" convex set of smaller volume and similar topological features to the previous set in this process. We stress that the two-dimensional sets in the optimal algorithm are rotated rectangles. What would be the proper sets in an arbitrary number of dimensions that would bound the non-convex sets resulting from the application of our general d-dimensional lemma?

2 Problem formulation

Given dimension $d \geq 2$, we define $D = [0, 1]^d$and the class F of functions, $f : D \to D$, that are Lipschitz continuous with constant 1 with respect to the infinity norm, i.e.,

$$\|f(x) - f(y)\| \leq \|x - y\|, \forall x, y \in D$$

where $\|\| = \|\|_\infty$henceforth. We seek an algorithm which, for every $f \in F$, computes a solution $\tilde{x} = \tilde{x}(f) \in D$ that satisfies the residual criterion

$$\|f(\tilde{x}) - \tilde{x}\| \leq \epsilon \tag{1}$$

where $0 < \epsilon < 0.5$. (If $\epsilon \geq 0.5$ then $x = (0.5, 0.5)$ satisfies [1]). The algorithm requires $n(f)$ function evaluations, where $n(f) \cong O(d \log \frac{1}{\epsilon})$. In the case of $d = 2$ the algorithm is based on Theorem 3.1 of [1], utilizes bisection of rectangles and envelope constructions, and has cost $2 \log_2 \frac{1}{\epsilon}$. Here we present a generalization of this theorem to the case of $d \geq 3$. We believe that the general result will provide the basis for construction of a future algorithm having the desired efficiency. So far we have been unable to construct such an algorithm. We stress that computing x_ϵ,$\|x_\epsilon - \alpha\| \leq \epsilon$, an ϵ-absolute approximation to the fixed point α, in the class of expanding functions is of infinite complexity in the worst case [3].

3 Definitions

For a given $f \in F$ and $i = 1, \ldots, d$ we define the fixed point sets F_i such that for each i,

$$F_i(f) = \{x \in D : f_i(x) = x_i\}.$$

We define $F(f) = \cap_{i=1}^d F_i(f)$, the nonempty set of all fixed points of f. For all $x \in \mathbb{R}^d$, $i = 1, \ldots, d$, and $s \in \{-1, 1\}$ we define the "open-ended" pyramid sets

$$A_i^s(x) = \{y \in \mathbb{R}^d : \|y - x\| = s(y_i - x_i)\}.$$

For all $x \in \mathbb{R}^d$, $i = 1, \ldots, d$, $s \in \{-1, 1\}$, and $c > 0$, we also define the "flat-top" pyramid set

$$Q_i^s(x, c) = \cup\{A_i^s(y) : y \in \mathbb{R}^d, \|y - x\| < c\}.$$

4 Constructive Lemma

In this section we prove our constructive lemma. It is a generalization of Theorem 3.1 of [1] to an arbitrary number of dimensions $d \geq 3$.

Lemma 4.1

For any $f \in F$, $i = 1, \ldots, d$, we let $x \in D$ be such that $f_i(x) \neq x_i$. Then the following holds:

(i) If $f_i(x) > x_i$ then $Q_i^{-1}(x, (f_i(x) - x_i)/2) \cap D \cap F_i(f) = \emptyset$.

(ii) If $f_i(x) < x_i$ then $Q_i^1(x, (x_i - f_i(x))/2) \cap D \cap F_i(f) = \emptyset$.

Proof. To show (i) we take any y such that $\|y - x\| < (f_i(x) - x_i)/2$, and $z \in A_i^{-1}(y) \cap D$. Then

$$|f_i(z) - f_i(y)| \leq \|f(z) - f(y)\| \leq \|z - y\| = y_i - z_i$$

and

$$\begin{aligned} f_i(y) - y_i = f_i(x) - (f_i(x) - f_i(y)) - x_i - (y_i - x_i) &\geq f_i(x) - x_i - 2\|y - x\| \\ &> f_i(x) - x_i - (f_i(x) - x_i) = 0, \end{aligned}$$

which implies

$$f_i(z) = f_i(y) + (f_i(z) - f_i(y)) > y_i - (y_i - z_i) = z_i.$$

To show (ii) we take any y such that $\|y - x\| < (x_i - f(x_i))/2$, and $z \in A_i^1(y) \cap D$. Then

$$|f_i(z) - f_i(y)| \leq \|f(z) - f(y)\| \leq \|z - y\| = z_i - y_i$$

and

$$\begin{aligned} f_i(y) - y_i = f_i(x) + (f_i(y) - f_i(x)) - x_i + (x_i - y_i) &\leq f_i(x) - x_i + 2\|y - x\| \\ &< f_i(x) - x_i + (x_i - f_i(x)) = 0, \end{aligned}$$

which implies

$$f_i(z) = f_i(y) + (f_i(z) - f_i(y)) < y_i + (z_i - y_i) = z_i. \quad \blacksquare$$

Comments

The above Lemma 4.1 states that after evaluating f at x we can remove from the original domain D the "flat-top" pyramid sets $Q_i^s(x, c_i)$ for all i such that $c_i = |f(x_i) - x_i|/2$ are not zero, since they do ***not*** contain fixed points of f_i, implying that they do not contain any fixed point of f as well. If this happens for all $i = 1, \ldots, d$ then we can reduce the volume of the set containing fixed points by a factor of at least two.

Open problems

The main obstacle in constructing a recursive algorithm (for $d \geq 3$) based on Lemma 4.1 is our apparent inability to construct a sequence of sets S_j that each contain a fixed point, are topologically "similar", decrease in volume, and are easy to represent, and then evaluating f at the "centers" of S_j. Also, it needs to be decided which sets can be removed from S_j in the case where $f_i(x) - x_i = 0$, i.e., when the current evaluation point x is a fixed point of some components of f.

We believe that by solving those problems we can obtain an optimal $O(d \log \frac{1}{\epsilon})$ cost algorithm for finding ϵ-residual solutions to the fixed points of functions in our class. We hope to address these issues in a future paper.

References

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